

# On Correlation Functions of Vertex Operator Algebras Associated to Jordan Algebras

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## 1 Introduction

Given a  $\mathbb{Z}_{\geq 0}$  graded vertex operator algebra (VOA)  $V = \bigoplus_{i=0}^{\infty} V_i$  with  $\dim(V_0) = 1$ , then  $V_1$  has a structure of Lie algebra, with the operation given by  $[a, b] = a(0)b$ ,  $\forall a, b \in V_1$ . The affine vertex algebra  $V_r(\mathfrak{g})$  of level  $r$  provides such an example, with  $V_1 = \mathfrak{g}$  as a Lie algebra.

When  $\dim(V_0) = 1$ ,  $\dim(V_1) = 0$ , then  $V_2$  has a structure of commutative (but not necessarily associative) algebra, with the operation  $a \circ b = a(1)b$ . For moonshine VOA  $V^\natural$  [FLM89] with  $\dim(V_0^\natural) = 1$ ,  $\dim(V_1^\natural) = 0$ , the corresponding commutative algebra  $V_2^\natural$  is the Griess algebra introduced by Griess in the study of the monster group [Gri82]. We call  $V_2$  of a VOA  $V$  with  $\dim(V_0) = 1$ ,  $\dim(V_1) = 0$  the Griess algebra of  $V$ . In [HL96] and [HL99], Lam constructed vertex algebras whose Griess algebras are simple Jordan algebras of type  $A, B, C$ . In [AM09], Ashihara and Miyamoto constructed for a Jordan algebra  $\mathcal{J}$  of Type  $B$ , a family of VOAs  $V_{\mathcal{J}, r}$  parameterized by a complex number  $r$ , such that  $(V_{\mathcal{J}, r})_0 = \mathbb{C}1$ ,  $(V_{\mathcal{J}, r})_1 = \{0\}$ , and

$$(V_{\mathcal{J}, r})_2 \cong \mathcal{J}$$

as Jordan algebra, and Lam's example for type  $B$  Jordan algebra is a quotient of  $V_{\mathcal{J}, 1}$ . The main result of this paper is a formula about the genus zero correlation function of generating fields in  $V_{\mathcal{J}, r}$ .

A. Albert classified finite dimensional simple Jordan algebras over an algebraic closed field  $\mathbb{F}$  with  $\text{char}(\mathbb{F}) = 0$  [Alb47]. We have a brief review of simple Jordan algebra of type  $B$ , and we only consider the case  $\mathbb{F} = \mathbb{C}$ . Let  $(\mathfrak{h}, (\cdot, \cdot))$  be a finite dimensional vector space with a non-degenerate symmetric bilinear form  $(\cdot, \cdot)$  and  $\dim(\mathfrak{h}) = d$ . Then  $\mathfrak{h} \otimes \mathfrak{h}$  has an associative algebra structure:

$$(a \otimes b)(u \otimes v) = (b, u)a \otimes v,$$

which induces a Jordan algebra structure on  $\mathfrak{h} \otimes \mathfrak{h}$ :

$$x \circ y = \frac{1}{2}(xy + yx), \quad \forall x, y \in \mathfrak{h} \otimes \mathfrak{h}.$$

Let  $\mathcal{J}$  be the Jordan subalgebra of  $\mathfrak{h} \otimes \mathfrak{h}$  consists of symmetric tensors:

$$\mathcal{J} \stackrel{\text{def}}{=} \text{span}\{L_{a,b} | a, b \in \mathfrak{h}\}, \quad L_{a,b} \stackrel{\text{def}}{=} a \otimes b + b \otimes a.$$

Then  $\mathcal{J}$  is essentially the Jordan algebra of symmetric  $d \times d$  matrices over  $\mathbb{C}$ , which is called simple Jordan algebra of type  $B$  according to Jacobson's notation [JJ49].

To state our result, we need to introduce some notations. Let  $L_{a,b}(z)$  denote the vertex operator associated to  $L_{a,b}$ . Given  $n$  vertex operators  $L_{a_1,b_1}(z_1), \dots, L_{a_n,b_n}(z_n)$ , we have a corresponding sequence  $T = (a_1, b_1) \cdots (a_n, b_n)$ . Recall that a derangement of the  $n$ -element set  $\{(a_1, b_1), \dots, (a_n, b_n)\}$  is a permutation of  $n$ -elements without fix point [FS09]. Let  $DR(T)$  denote the set of all derangements corresponding to  $T$ . For  $\sigma \in DR(T)$  we decompose it as disjoint cycles

$$\sigma = (C_1) \cdots (C_s),$$

and we use  $\sigma(i)$  to denote the image of each label  $i$  under the action of  $\sigma$ . We also use the symbol

$$c(\sigma) \stackrel{\text{def}}{=} s$$

to denote the number of disjoint cycles in  $\sigma$ .

Our main result is stated as follows:

**Theorem 1.** *Given  $n$  vertex operators  $L_{a_1,b_1}(z_1), \dots, L_{a_n,b_n}(z_n)$ , and  $T = (a_1, b_1) \cdots (a_n, b_n)$ . Then the corresponding genus zero correlation function is given by*

$$\langle 1', L_{a_1,b_1}(z_1) \cdots L_{a_n,b_n}(z_n) \cdot 1 \rangle = \sum_{\sigma \in DR(T)} \Gamma(\sigma, T) \Gamma(\sigma; Z) r^{c(\sigma)}, \quad (1)$$

where the symbols are described as follows: Assume that  $\sigma = (C_1) \cdots (C_s) = (k_{11} \cdots k_{1t_1}) \cdots (k_{s1} \cdots k_{st_s})$ , then

$$\begin{aligned} \Gamma(\sigma, T) &\stackrel{\text{def}}{=} 2^{-s-n} \prod_{i=1}^s \text{Tr}(L_{a_{k_{i1}}, b_{k_{i1}}} \cdots L_{a_{k_{it_i}}, b_{k_{it_i}}}), \\ \Gamma(\sigma; Z) &\stackrel{\text{def}}{=} \prod_{i=1}^n \frac{1}{(z_i - z_{\sigma(i)})^2}, \end{aligned}$$

where  $\text{Tr}(a \otimes b)$  is the trace of  $a \otimes b \in \mathfrak{h} \otimes \mathfrak{h}$  given by:

$$\text{Tr}(a \otimes b) = (a, b).$$

We have a more general formula about the “correlation function” of some two variable formal power series, which will be shown in Proposition 2, and we prove Theorem 1 as a corollary of this proposition.

We give an example of Theorem 1 for the case  $n = 4$ . The sequence is  $T = (a_1, b_1) \cdots (a_4, b_4)$ , and there are 9 elements in  $DR(T)$ , given by:

$$(12)(34), (13)(24), (14)(23); (1234), (1243), (1324), (1342), (1423), (1432).$$

Then Theorem 1 gives

$$\begin{aligned} &\langle 1', L_{a_1,b_1}(z_1) L_{a_2,b_2}(z_2) L_{a_3,b_3}(z_3) L_{a_4,b_4}(z_4) 1 \rangle \\ &= \frac{\text{Tr}(L_{a_1,b_1} L_{a_2,b_2}) \text{Tr}(L_{a_3,b_3} L_{a_4,b_4}) r^2}{64(z_1 - z_2)^4 (z_3 - z_4)^4} \\ &\quad + \frac{\text{Tr}(L_{a_1,b_1} L_{a_3,b_3}) \text{Tr}(L_{a_2,b_2} L_{a_4,b_4}) r^2}{64(z_1 - z_3)^4 (z_2 - z_4)^4} + \frac{\text{Tr}(L_{a_1,b_1} L_{a_4,b_4}) \text{Tr}(L_{a_2,b_2} L_{a_3,b_3}) r^2}{64(z_1 - z_4)^4 (z_2 - z_3)^4} \end{aligned}$$

$$\begin{aligned}
& + \frac{\text{Tr}(L_{a_1,b_1}L_{a_2,b_2}L_{a_3,b_3}L_{a_4,b_4})r}{32(z_1-z_2)^2(z_2-z_3)^2(z_3-z_4)^2(z_4-z_1)^2} + \frac{\text{Tr}(L_{a_1,b_1}L_{a_2,b_2}L_{a_4,b_4}L_{a_3,b_3})r}{32(z_1-z_2)^2(z_2-z_4)^2(z_4-z_3)^2(z_3-z_1)^2} \\
& + \frac{\text{Tr}(L_{a_1,b_1}L_{a_3,b_3}L_{a_2,b_2}L_{a_4,b_4})r}{32(z_1-z_3)^2(z_3-z_2)^2(z_2-z_4)^2(z_4-z_1)^2} + \frac{\text{Tr}(L_{a_1,b_1}L_{a_3,b_3}L_{a_4,b_4}L_{a_2,b_2})r}{32(z_1-z_3)^2(z_3-z_4)^2(z_4-z_2)^2(z_2-z_1)^2} \\
& + \frac{\text{Tr}(L_{a_1,b_1}L_{a_4,b_4}L_{a_2,b_2}L_{a_3,b_3})r}{32(z_1-z_4)^2(z_4-z_2)^2(z_2-z_3)^2(z_3-z_1)^2} + \frac{\text{Tr}(L_{a_1,b_1}L_{a_4,b_4}L_{a_3,b_3}L_{a_2,b_2})r}{32(z_1-z_4)^2(z_4-z_3)^2(z_3-z_2)^2(z_2-z_1)^2}.
\end{aligned}$$

When  $r = 1$ , the correlation function in Theorem 1 is the same as correlation function of vertex operators  $\frac{1}{2} : a_1(z_1)b_1(z_1) : , \dots , \frac{1}{2} : a_n(z_n)b_n(z_n) :$  in Heisenberg VOA  $S(\hat{\mathfrak{h}}_-)$ . The theorem can be proved using Wick's theorem (see Theorem 3.3 in [Kac98]).

When  $\dim(\mathcal{J}) = 1$ , the VOA  $V_{\mathcal{J},r}$  is the same as the VOA associated to Virasoro algebra with central charge equal to  $r$ , and the conformal vector is  $\omega = L_{e,e}$ ,  $(e,e) = 1$ . In this case our formula agrees with the result in [HT12] (Theorem 2.3), where the genus zero correlation function for Virasoro field is computed:

$$\langle 1', \omega(z_1) \cdots \omega(z_n) \cdot 1 \rangle = \sum_{\sigma \in DR(T)} \Gamma(\sigma; Z) \left(\frac{r}{2}\right)^{c(\sigma)} = \sum_{\sigma \in DR(T)} \left(\prod_{i=1}^n \frac{1}{(z_i - z_{\sigma(i)})^2}\right) \left(\frac{r}{2}\right)^{c(\sigma)}.$$

When  $n = 4$ , the result is:

$$\begin{aligned}
& \langle 1', \omega(z_1)\omega(z_2)\omega(z_3)\omega(z_4)1 \rangle \\
& = \frac{r^2}{4(z_1-z_2)^4(z_3-z_4)^4} + \frac{r^2}{4(z_1-z_3)^4(z_2-z_4)^4} + \frac{r^2}{4(z_1-z_4)^4(z_2-z_3)^4} \\
& + \frac{r}{2(z_1-z_2)^2(z_2-z_3)^2(z_3-z_4)^2(z_4-z_1)^2} + \frac{r}{2(z_1-z_2)^2(z_2-z_4)^2(z_4-z_3)^2(z_3-z_1)^2} \\
& + \frac{r}{2(z_1-z_3)^2(z_3-z_2)^2(z_2-z_4)^2(z_4-z_1)^2} + \frac{r}{2(z_1-z_3)^2(z_3-z_4)^2(z_4-z_2)^2(z_2-z_1)^2} \\
& + \frac{r}{2(z_1-z_4)^2(z_4-z_2)^2(z_2-z_3)^2(z_3-z_1)^2} + \frac{r}{2(z_1-z_4)^2(z_4-z_3)^2(z_3-z_2)^2(z_2-z_1)^2}.
\end{aligned}$$

Theorem 1 can also be viewed as an analogue of Theorem 2.3.1 in [FZ92], where the genus zero correlation function of the generating fields in  $V_r(\mathfrak{g})$  is computed:

**Theorem 2.** *The genus zero correlation function of fields  $a_1(z_1), \dots, a_n(z_n)$ ,  $a_i \in \mathfrak{g}$  is given by the formula:*

$$\langle 1', a_1(z_1) \cdots a_n(z_n) 1 \rangle = \sum_{\sigma=(C_1) \cdots (C_s) \in DR(T)} \prod_{i=1}^s f_{C_i}(a_1, \dots, a_n; z_1, \dots, z_n) (-r)^s,$$

where

$$f_{C_i}(a_1, \dots, a_n; z_1, \dots, z_n) \stackrel{\text{def}}{=} \frac{\text{Tr}(a_{k_1} \cdots a_{k_t})}{(z_{k_1} - z_{k_2}) \cdots (z_{k_{t-1}} - z_{k_t})(z_{k_t} - z_{k_1})}.$$

The trace here is normalized so that  $\forall x, y \in \mathfrak{g}$ ,  $\text{Tr}(xy)$  equals to the normalized Killing form of  $\mathfrak{g}$ .

The content of this paper is organized as follows: In section 2 we have an overview of the construction in [AM09] and some results in [NS10], then we introduce some two variable formal power series whose commutation relations are computed. In section 3, we introduce diagrams and some other related notations, and section 4 is devoted to the proof of Theorem 1.

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## 2 Vertex Operator Algebras Associated to Jordan Algebras of Type $B$

In this section, first we give a brief overview of the vertex operator algebra  $V_{\mathcal{J},r}$  constructed in [AM09]. Then, we introduce two-variable formal power series  $L_{a,b}(z, w), L_{a,b}^{\pm\pm}(z, w)$  and compute some commutation relations.

Let  $\mathfrak{h}$  be a finite dimensional vector space with a non-degenerate symmetric bilinear form  $(\cdot, \cdot)$ . Assume  $\dim(\mathfrak{h}) = d$ , then the Heisenberg Lie algebra associated to  $(\mathfrak{h}, (\cdot, \cdot))$  is:

$$\hat{\mathfrak{h}} = L(\mathfrak{h}) \oplus \mathbb{C}c, \quad L(\mathfrak{h}) = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}],$$

with the Lie bracket given by:

$$[a(m), b(n)] = m(a, b)\delta_{m+n}c, \quad [x, c] = 0, \quad \forall x \in \hat{\mathfrak{h}},$$

where  $a(m) = at^m \in L(\mathfrak{h})$ . Noting that

$$\hat{\mathfrak{h}}_- = \mathfrak{h} \otimes \mathbb{C}t^{-1}[t^{-1}]$$

is a commutative Lie subalgebra, and the Fock space  $S(\hat{\mathfrak{h}}_-) = U(\hat{\mathfrak{h}}_-)$  has a vertex operator algebra structure [FLM89].

Take the following subspace  $\mathfrak{L}$  of the universal enveloping algebra  $U(\hat{\mathfrak{h}})$  spanned by quadratic elements:

$$\mathfrak{L} \stackrel{def}{=} \text{span}\{a(m)b(n) | a, b \in \mathfrak{h}, m, n \in \mathbb{Z}\}.$$

Since  $[a(m), b(n)] = m(a, b)\delta_{m+n}c$  so  $c \in \mathfrak{L}$ .

Define a new Lie bracket  $[x, y]_{new}$  on  $\mathfrak{L}$  by

$$[x, y]_{new} \stackrel{def}{=} \frac{1}{c}[x, y], \quad \forall x, y \in \mathfrak{L}. \quad (2)$$

The identity (3) below shows that the right hand side is still in  $\mathfrak{L}$ .

For convenience we use the notation:

$$L_{a,b}(m, n) \stackrel{def}{=} \frac{1}{2} : a(m)b(n) :,$$

where

$$: a(m)b(n) := \begin{cases} b(n)a(m), & m \geq n, \\ a(m)b(n), & \text{else.} \end{cases}$$

Let

$$\mathfrak{B} \stackrel{def}{=} \text{span}\{L_{a,b}(m,n) | a, b \in \mathfrak{h}, m, n \in \mathbb{Z}\}.$$

Noting that we have a split of  $\mathfrak{L}$ :

$$\mathfrak{L} = \mathfrak{B} \oplus \mathbb{C}c.$$

For two elements  $L_{a,b}(m,n), L_{u,v}(p,q) \in \mathfrak{B}$ , we have

$$\begin{aligned} & [L_{a,b}(m,n), L_{u,v}(p,q)]_{new} \\ &= \frac{1}{4c} [a(m)b(n), u(p)v(q)] \\ &= \frac{1}{4c} [a(m)b(n), u(p)]v(q) + \frac{1}{4c} u(p)[a(m)b(n), v(q)] \\ &= \frac{1}{4c} a(m)[b(n), u(p)]v(q) + \frac{1}{4c} [a(m), u(p)]b(n)v(q) + \frac{1}{4c} u(p)a(m)[b(n), v(q)] + \frac{1}{4c} u(p)[a(m), v(q)]b(n) \\ &= \frac{1}{4} n\delta_{n+p}(b, u)a(m)v(q) + \frac{1}{4} m\delta_{m+p}(a, u)b(n)v(q) \\ & \quad + \frac{1}{4} n\delta_{n+q}(b, v)u(p)a(m) + \frac{1}{4} m\delta_{m+q}(a, v)u(p)b(n) \in \mathfrak{L}. \end{aligned} \tag{3}$$

Decompose  $\mathfrak{B}$  into:

$$\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-,$$

where

$$\mathfrak{B}_- \stackrel{def}{=} \text{span}\{L_{a,b}(m,n) | m, n < 0\}, \quad \mathfrak{B}_+ \stackrel{def}{=} \text{span}\{L_{a,b}(m,n) | n \geq 0 \text{ or } m \geq 0\}.$$

Then we have a decomposition of  $\mathfrak{L}$ :

$$\mathfrak{L}_- = \mathfrak{B}_-, \quad \mathfrak{L}_+ = \mathfrak{B}_+ \oplus \mathbb{C}c.$$

Define a 1-dimensional  $\mathfrak{L}_+$  module  $\mathbb{C}1$  :

$$x1 = 0, \quad \forall x \in \mathfrak{B}_+, \quad c1 = r1.$$

Then by induction from  $U(\mathfrak{L}_+)$  to  $U(\mathfrak{L})$ , we have a  $U(\mathfrak{L})$  module  $M_r$ :

$$\begin{aligned} M_r & \stackrel{def}{=} U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_+)} \mathbb{C}1 \\ & \cong U(\mathfrak{L}_-)1 \\ & = \text{span}\{L_{a_1, b_1}(-m_1, -n_1) \cdots L_{a_k, b_k}(-m_k, -n_k) \cdot 1 | m_i, n_i \in \mathbb{Z}_{\geq 1}, a_i, b_i \in \mathfrak{h}\}. \end{aligned}$$

For  $a, b \in \mathfrak{h}$  define the following operator  $L_{a,b}(l)$ :

$$L_{a,b}(l) \stackrel{def}{=} \sum_{k \in \mathbb{Z}} L_{a,b}(-k + l - 1, k),$$

and field  $L_{a,b}(z)$  by:

$$L_{a,b}(z) \stackrel{def}{=} \sum_{l \in \mathbb{Z}} L_{a,b}(l) z^{-l-1}.$$

It is proved in [AM09] that these fields are mutually local.

So these mutually local fields generate a vertex algebra, which is denoted by  $V_{\mathcal{J},r}$ :

$$V_{\mathcal{J},r} = \text{span}\{L_{a_1,b_1}(m_1) \cdots L_{a_k,b_k}(m_k) \cdot 1 \mid m_i \in \mathbb{Z}, a_i, b_i \in \mathfrak{h}\}.$$

It is proved in [NS10] that  $V_{\mathcal{J},r} = M_r$  holds, the Virasoro element of  $V_{\mathcal{J},r}$  is

$$\omega = \sum_k L_{e_k, e_k}(-1, -1) \cdot 1,$$

where  $\{e_k\}$  is an orthonormal basis of  $\mathfrak{h}$ , and  $\omega(1)$  gives a graduation of  $V_{\mathcal{J},r}$ :

$$V_{\mathcal{J},r} = \sum_{k \geq 0} (V_{\mathcal{J},r})_k.$$

The Griess algebra  $(V_{\mathcal{J},r})_2$  is isomorphic to  $\mathcal{J}$ :

$$L_{a,b}(-1, -1) \cdot 1 \mapsto L_{a,b} \stackrel{def}{=} a \otimes b + b \otimes a.$$

We introduce two-variable formal power series  $L_{a,b}^{--}(z, w)$ ,  $L_{a,b}^{-+}(z, w)$ ,  $L_{a,b}^{+-}(z, w)$ ,  $L_{a,b}^{++}(z, w)$ , and  $L_{a,b}(z, w)$  whose coefficients are in  $\mathfrak{L}$ :

$$\begin{aligned} L_{a,b}(z, w) &\stackrel{def}{=} \sum_{m,n \in \mathbb{Z}} L_{a,b}(m, n) z^{-m-1} w^{-n-1}, \\ L_{a,b}^{--}(z, w) &\stackrel{def}{=} \sum_{m,n < 0} L_{a,b}(m, n) z^{-m-1} w^{-n-1}, \quad L_{a,b}^{-+}(z, w) \stackrel{def}{=} \sum_{m < 0, n \geq 0} L_{a,b}(m, n) z^{-m-1} w^{-n-1}, \\ L_{a,b}^{+-}(z, w) &\stackrel{def}{=} \sum_{m \geq 0, n < 0} L_{a,b}(m, n) z^{-m-1} w^{-n-1}, \quad L_{a,b}^{++}(z, w) \stackrel{def}{=} \sum_{m,n \geq 0} L_{a,b}(m, n) z^{-m-1} w^{-n-1}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} L_{a,b}^{--}(z, w) &= L_{b,a}^{--}(w, z), \quad L_{a,b}^{++}(z, w) = L_{b,a}^{++}(w, z), \quad L_{a,b}^{+-}(z, w) = L_{b,a}^{-+}(w, z), \\ L_{a,b}(z, w) &= L_{a,b}^{--}(z, w) + L_{a,b}^{-+}(z, w) + L_{a,b}^{+-}(z, w) + L_{a,b}^{++}(z, w). \end{aligned} \tag{4}$$

and

$$L_{a,b}(z) = L_{a,b}(z, z)$$

as fields on  $V_{\mathcal{J},r}$ .

Then we have the following closed formula for the commutation relations that we need later:

**Proposition 1.** *As formal power series with coefficients taking value in  $\text{End}(V_{\mathcal{J},r})$ , we have*

$$\begin{aligned} [L_{a,b}^{++}(x,y), L_{u,v}^{--}(z,w)] &= \frac{1}{2}(b,v)\iota_{y,w}(y-w)^{-2}L_{u,a}^{++}(z,x) + \frac{1}{2}(a,v)\iota_{x,w}(x-w)^{-2}L_{u,b}^{--}(z,y) \\ &\quad + \frac{1}{2}(a,u)\iota_{x,z}(x-z)^{-2}L_{v,b}^{++}(w,y) + \frac{1}{2}(b,u)\iota_{y,z}(y-z)^{-2}L_{v,a}^{--}(w,x) \\ &\quad + \frac{1}{4}r(a,u)(b,v)\iota_{x,z}(x-z)^{-2}\iota_{y,w}(y-w)^{-2} \\ &\quad + \frac{1}{4}r(a,v)(b,u)\iota_{x,w}(x-w)^{-2}\iota_{y,z}(y-z)^{-2}, \end{aligned} \quad (5)$$

$$\begin{aligned} [L_{a,b}^{++}(x,y), L_{u,v}^{+-}(z,w)] &= \frac{1}{2}(a,u)L_{b,v}^{++}(y,w)\iota_{x,z}(x-z)^{-2} + \frac{1}{2}(b,u)L_{a,v}^{++}(x,w)\iota_{y,z}(y-z)^{-2}, \quad (6) \\ [L_{a,b}^{+-}(x,y), L_{u,v}^{++}(z,w)] &= 0. \end{aligned}$$

**Proof.** We only prove (5), and others are obtained in a similar way. Consider the following formal power series with coefficients in  $U(\hat{\mathfrak{h}})$

$$\begin{aligned} a_-(z) &\stackrel{\text{def}}{=} \sum_{k < 0} a(m)z^{-k-1}, \quad a_+(z) \stackrel{\text{def}}{=} \sum_{k \geq 0} a(m)z^{-k-1}, \\ a(z) &\stackrel{\text{def}}{=} \sum_k a(m)z^{-k-1} = a_-(z) + a_+(z), \quad \forall a \in \mathfrak{h}. \end{aligned}$$

A direct computation shows that

$$[a_+(z), b_-(w)] = (a,b)c\iota_{z,w} \frac{1}{(z-w)^2}. \quad (7)$$

We view  $a_{\pm}(x)b_{\pm}(y)$  as formal power series with coefficients in  $\mathfrak{L}$ , then we have

$$\begin{aligned} L_{a,b}^{++}(z,w) &= \frac{1}{2}a_+(z)b_+(w), \quad L_{a,b}^{+-}(z,w) = \frac{1}{2}a_-(z)b_+(w), \\ L_{a,b}^{+-}(z,w) &= \frac{1}{2}b_-(w)a_+(z), \quad L_{a,b}^{--}(z,w) = \frac{1}{2}a_-(z)b_-(w). \end{aligned}$$

So 5 is proved by the following computation:

$$\begin{aligned} &[a_+(x)b_+(y), u_-(z)v_-(w)]_{n \in w} \\ &= \frac{1}{c}[a_+(x)b_+(y), u_-(z)v_-(w)] \\ &= (b,v)\iota_{y,w}(y-w)^{-2}u_-(z)a_+(x) + (a,v)\iota_{x,w}(x-w)^{-2}u_-(z)b_+(y) \\ &\quad + (a,u)\iota_{x,z}(x-z)^{-2}b_+(y)v_-(w) + (b,u)\iota_{y,z}(y-z)^{-2}a_+(x)v_-(w) \\ &= (b,v)\iota_{y,w}(y-w)^{-2}u_-(z)a_+(x) + (a,v)\iota_{x,w}(x-w)^{-2}u_-(z)b_+(y) \\ &\quad + (a,u)\iota_{x,z}(x-z)^{-2}v_-(w)b_+(y) + (b,u)\iota_{y,z}(y-z)^{-2}v_-(w)a_+(x) \\ &\quad + (a,u)(b,v)c\iota_{x,z}(x-z)^{-2}\iota_{y,w}(y-w)^{-2} \\ &\quad + (a,v)(b,u)c\iota_{x,w}(x-w)^{-2}\iota_{y,z}(y-z)^{-2}. \end{aligned}$$

### 3 Diagrams, Derangements, and Some Necessary Notations

In this section we introduce a notation called diagram over a sequence  $T = (a_1, b_1) \cdots (a_n, b_n)$ , denoted by  $D(T)$ , and it will be shown that there is a surjective map from  $D(T)$  to  $DR(T)$ . We also introduce the notation of diagram over  $T$  compatible with sign  $\epsilon$ , denoted by  $D(T^\epsilon)$ , and the two obvious operations of diagram turn out to have correspondences with the commutation relations, which will be used to prove Theorem 1.

Let  $1'$  be the unique element in  $V_{\mathcal{J},r}^*$  satisfying  $\langle 1', 1 \rangle = 1$ ,  $\langle 1', v \rangle = 0, \forall v \in (V_{\mathcal{J},r})_{i \geq 1}$ . Then the genus zero correlation function of  $L_{a_1, b_1}(z_1), \dots, L_{a_n, b_n}(z_n)$  given by

$$\langle 1', L_{a_1, b_1}(z_1) \cdots L_{a_n, b_n}(z_n) 1 \rangle,$$

which is a power series in  $\mathbb{C}[[z_1, z_1^{-1} \cdots z_n, z_n^{-1}]]$ . According to [FLM89], this formal power series converges to a rational function in the domain  $|z_1| > \cdots > |z_n|$ .

Given the sequence  $T = (a_1, b_1) \cdots (a_n, b_n)$ , a sign on  $T$  is a function  $\epsilon : \{a_1, b_1, \dots, a_n, b_n\} \rightarrow \{+, -\}$ . If  $\epsilon(a_i) = \epsilon_i, \epsilon(b_i) = \delta_i$ , we also write

$$T^\epsilon = (a_1^{\epsilon_1}, b_1^{\delta_1}) \cdots (a_n^{\epsilon_n}, b_n^{\delta_n}),$$

and we call  $(a_i^{\epsilon_i}, b_i^{\delta_i})$  the  $i$ -th pair of  $T^\epsilon$ .

A diagram over the sequence  $T = (a_1, b_1) \cdots (a_n, b_n)$  is a graph, with vertex set  $V = \{a_1, b_1, \dots, a_n, b_n\}$ , and edge set  $E$  consists of unordered pairs  $\{u, v\}$ ,  $u, v \in V$  satisfying:

- (1).  $|E| = n$ .
- (2).  $\{a_i, b_i\}, \{a_i, a_i\}, \{b_i, b_i\} \notin E, \forall i = 1, \dots, n$ .
- (3). Any two edges have no common point.

Denote the set of all diagrams over  $T$  by  $D(T)$ . Observing that such a graph is always with  $2n$  vertices and  $n$  disjoint edges.

A diagram over the signed sequence  $T^\epsilon = (a_1^{\epsilon_1}, b_1^{\delta_1}) \cdots (a_n^{\epsilon_n}, b_n^{\delta_n})$  is a diagram over  $T$  compatible with the sign  $\epsilon$  in the following sense:

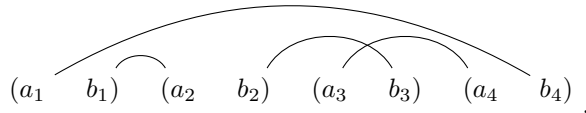
- (4).  $\forall \{u, v\} \in E$ , if  $u = a_i$  or  $b_i$ ,  $v = a_j$  or  $b_j$ ,  $i < j$ , then  $\epsilon(u) = +, \epsilon(v) = -$ .

Denote the set of all diagrams over  $T$  which are compatible with  $\epsilon$  by  $D(T^\epsilon)$ . It may happen that for some  $\epsilon$ ,  $D(T^\epsilon) = \emptyset$ . For example, if  $T^\epsilon = (a_1^+, b_1^+)(a_2^-, b_2^-)(a_3^-, b_3^-)(a_4^+, b_4^+)$ , then  $D(T^\epsilon) = \emptyset$ .

Given a diagram in  $\coprod_\epsilon D(T^\epsilon)$ , we can get a diagram in  $D(T)$  by forgetting the sign  $\epsilon$ , and conversely, for a diagram  $D \in D(T)$ , there is a unique way of assigning a sign  $\epsilon$  satisfying the compatibility condition. So we have

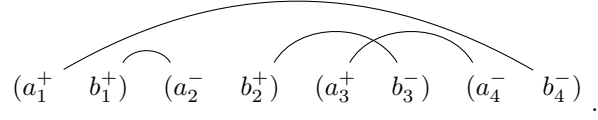
$$D(T) = \coprod_\epsilon D(T^\epsilon). \quad (8)$$

It will be helpful illustrating  $D(T)$  or  $D(T^\epsilon)$  using graphs. We give some examples and non-examples of  $D(T)$  and  $D(T^\epsilon)$ . Let  $T = (a_1, b_1)(a_2, b_2)(a_3, b_3)(a_4, b_4)$ . Then the following is a diagram over  $T$

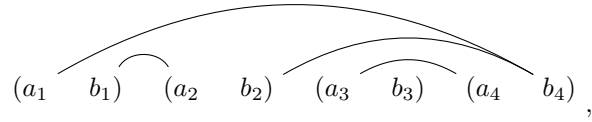
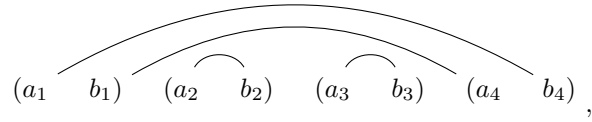
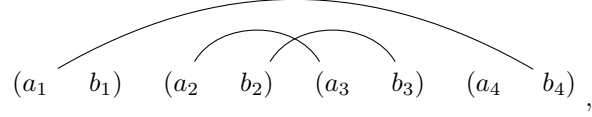




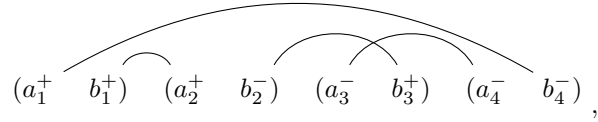
And there is a unique way of adding the sign:



But the followings are not diagrams over  $T$ :



because they violate the conditions (1),(2), and (3) in the definition respectively. The following

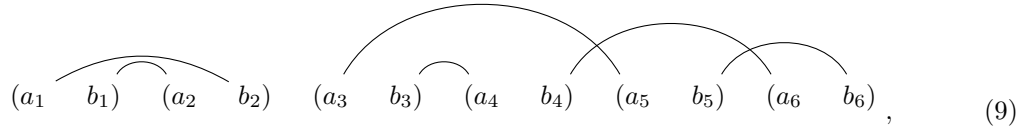


is a diagram over  $T$ , but it is not compatible with the sign  $\epsilon = (++)(+)(-+)(-+)(--)$ .

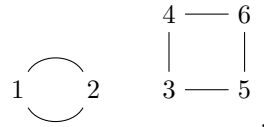
We will establish a map from  $D(T)$  to  $DR(T)$  in two steps.

First, for a diagram  $D = \{V, E\} \in D(T)$ , we can get a new graph by identifying two vertices  $a_i, b_i$  in the same pair  $(a_i, b_i)$ ,  $\forall i = 1, \dots, n$ . The defining condition of  $D(T)$  implies that this graph is always disjoint unions of cycle graphs.

For example, when  $T = (a_1, b_1) \cdots (a_6, b_6)$  and  $D(T)$  is graphically represented by:

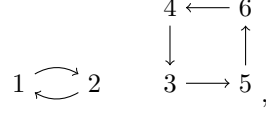


the corresponding graph is given by



Next, we add an orientation on each cycle to get a directed graph. For a cycle with vertices labelled by  $i_1, \dots, i_t$ , in which  $i_1$  be the smallest number, when  $t \geq 3$ , there are two possibilities adding the orientation. We choose the edge  $\{a_{i_1}, b_j\} \in E$  (or  $\{a_{i_1}, a_j\} \in E$ ), and take the orientation in the direction from  $i_1$  to  $j$ .

For example, for diagram (9), the corresponding directed graph is



where  $i_1 = 3, j = 5$ , in the second cycle. Noting that these graphs are essentially the same as what is called 'Virasoro graphs' in [HT12].

For every such directed graph we can get a corresponding element in  $DR(T)$  in the obvious way. For the above example, (12)(3564) is the corresponding derangement. We use  $\sigma_D$  to denote the derangement corresponding to  $D$ .

We have the following combinatorial lemma:

**Lemma 1.** *Given a sequence  $T = (a_1, b_1) \cdots (a_n, b_n), n \geq 2$ , the map defined above:*

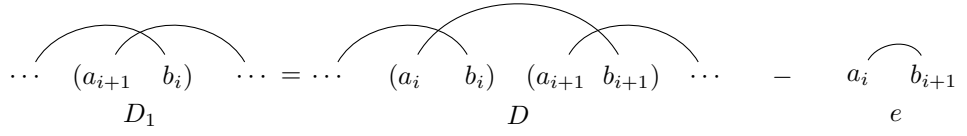
$$D \mapsto \sigma_D.$$

*is surjective. Further, for  $\sigma = (C_1), \dots, (C_s) \in DR(T)$ , there are exactly  $2^{n-s}$  diagrams  $D \in D(T)$  with  $\sigma_D = \sigma$ .*

We also consider some operations on the diagram. Suppose  $T = (a_1, b_1) \cdots (a_n, b_n)$ ,  $D = (V, E) \in D(T)$ ,  $e = \{a_i, b_{i+1}\} \in E$ , then we can form a new diagram  $D_1 = (V_1, E_1)$  by deleting  $e$ . More precisely the new sequence is:

$$T_1 = (a_1, b_1) \cdots (a_{i-1}, b_{i-1})(a_{i+1}, b_i)(a_{i+2}, b_{i+2}) \cdots (a_n, b_n),$$

and this is described graphically by:

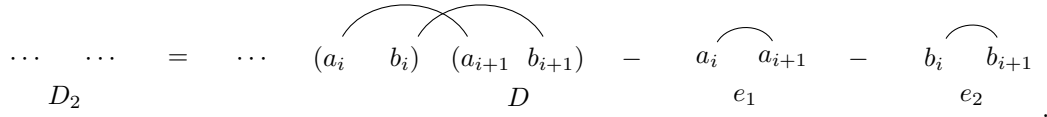


We simply write  $D_1 = D - e$ , or equivalently  $D = D_1 + e$ .

Similarly, if  $e_1 = \{a_i, a_{i+1}\}, e_2 = \{b_i, b_{i+1}\} \in E$ , we can form a new diagram  $D_2 = (V_2, E_2)$  by deleting both  $e_1, e_2$ :

$$T_2 = (a_1, b_1) \cdots (a_{i-1}, b_{i-1})(a_{i+2}, b_{i+2}) \cdots (a_n, b_n),$$

which is described graphically as



We also write  $D_2 = D - e_1 - e_2$  or  $D = D_2 + e_1 + e_2$ . Noting that  $D_2 = \emptyset$  may happen.

Later on we use capital letters  $Z, W$  to denote the set of variables in the formal power series  $L_{a_1, b_1}(z_1, w_1), \dots, L_{a_n, b_n}(z_n, w_n)$ :

$$Z = \{z_1, \dots, z_n\}, \quad W = \{w_1, \dots, w_n\}.$$

In (5) and (6) it is seen that  $z_i$  is always associated to  $a_i$  and  $w_i$  is associated to  $b_i$ .

We need to define some functions related to the graphical data. For a diagram  $D = (V, E)$  over  $T = (a_1, b_1) \dots (a_n, b_n)$  and for an edge  $e = \{a_i, b_j\} \in E$ , define

$$K(e; Z, W) \stackrel{\text{def}}{=} \frac{1}{(z_i - w_j)^2}, \quad Q(e; Z, W) \stackrel{\text{def}}{=} \frac{\frac{1}{2}(a_i, b_j)}{(z_i - w_j)^2},$$

and similarly for an edge  $e = \{a_i, a_j\}$ , define

$$K(e; Z, W) \stackrel{\text{def}}{=} \frac{1}{(z_i - z_j)^2}, \quad Q(e; Z, W) \stackrel{\text{def}}{=} \frac{\frac{1}{2}(a_i, a_j)}{(z_i - z_j)^2},$$

and for  $e = \{b_i, b_j\}$ :

$$K(e; Z, W) \stackrel{\text{def}}{=} \frac{1}{(w_i - w_j)^2}, \quad Q(e; Z, W) \stackrel{\text{def}}{=} \frac{\frac{1}{2}(b_i, b_j)}{(w_i - w_j)^2},$$

and we also define the functions  $\Gamma(D), R(D; Z, W)$ :

$$\Gamma(D) \stackrel{\text{def}}{=} \prod_{\{u, v\} \in E} (u, v), \quad R(D; Z, W) \stackrel{\text{def}}{=} r^{c(\sigma_D)} \prod_{e \in E} Q(e; Z, W).$$

Observing that  $R(D; Z, W)$  can also be written as

$$R(D; Z, W) \stackrel{\text{def}}{=} \Gamma(D) r^{c(\sigma_D)} \prod_{e \in E} K(e; Z, W), \tag{10}$$

through a direct computation. When  $D = \emptyset$  we define

$$R(\emptyset; Z, W) = 1$$

by convention.

## 4 Proof of Theorem 1

In this section we give the proof of Theorem 1. To compute the correlation function

$$\langle 1', L_{a_1, b_1}(z_1) \dots L_{a_n, b_n}(z_n) 1 \rangle,$$

we need to compute the following “correlation function” of formal power series:

$$\langle 1', L_{a_1, b_1}(z_1, w_1) \dots L_{a_n, b_n}(z_n, w_n) 1 \rangle.$$

By (4) we have

$$\langle 1', L_{a_1, b_1}(z_1, w_1) \cdots L_{a_n, b_n}(z_n, w_n) 1 \rangle = \sum_{\epsilon} \langle 1', L_{a_1, b_1}^{\epsilon_1, \delta_1}(z_1, w_1) \cdots L_{a_n, b_n}^{\epsilon_n, \delta_n}(z_n, w_n) 1 \rangle,$$

where  $\epsilon = (\epsilon_1, \delta_1) \cdots (\epsilon_n, \delta_n)$  runs over all  $4^n$  possible signs over  $T$ . We need to compute each term in the summation of the right hand side.

When  $D(T^\epsilon) = \emptyset$ , it is easy to see that

$$\langle 1', L_{a_1, b_1}^{\epsilon_1, \delta_1}(z_1, w_1) \cdots L_{a_n, b_n}^{\epsilon_n, \delta_n}(z_n, w_n) 1 \rangle = 0.$$

We have the following key lemma:

**Lemma 2.** For signed sequence  $T^\epsilon = (a_1^{\epsilon_1}, b_1^{\delta_1}) \cdots (a_n^{\epsilon_n}, b_n^{\delta_n})$ , we have

$$\langle 1', L_{a_1, b_1}^{\epsilon_1, \delta_1}(z_1, w_1), \cdots, L_{a_n, b_n}^{\epsilon_n, \delta_n}(z_n, w_n) 1 \rangle = \sum_{D \in D(T^\epsilon)} R(D; Z, W), \quad (11)$$

and the right hand side is 0 if  $D(T^\epsilon) = \emptyset$  by convention.

**Proof.** Let  $T^\epsilon(n, k)$  denote the set of signed sequence  $T^\epsilon$  which has  $n$  pairs, and there are  $k$  consecutive pairs of form  $(a^+, b^+)$  appear in the leftmost of  $T^\epsilon$ . For example, for

$$T^\epsilon = (a_1^+, b_1^+)(a_2^+, b_2^+)(a_3^-, b_3^+)(a_4^+, b_4^-)(a_5^-, b_5^-)(a_6^-, b_6^-),$$

then we have  $T^\epsilon \in T^\epsilon(6, 2)$ .

We prove this by induction on  $n + k$ . When  $n = k = 0$ , obviously  $T^\epsilon = \emptyset$  and both sides equal to 1; When  $k = 0, n > 0$ , or  $n < 2k$  it is also easy to see that both sides equal to 0.

So it is enough to show that the correctness for  $T^\epsilon \in T^\epsilon(n, k-1) \cup T^\epsilon(n-1, k) \cup T^\epsilon(n-1, k-1) \cup T^\epsilon(n-2, k-1)$  will imply the correctness for  $T^\epsilon \in T^\epsilon(n, k)$ , and we only need to consider the case when  $n \geq 1$  and  $n \geq 2k$ .

For  $T^\epsilon \in T^\epsilon(n, k)$ , suppose  $T^\epsilon = \cdots (a_k^+, b_k^+) \cdots$ , then there are two cases:

**Case 1** The signed sequence  $T^\epsilon = \cdots (a_k^+, b_k^+)(a_{k+1}^-, b_{k+1}^+) \cdots$ , or  $\cdots (a_k^+, b_k^+)(a_{k+1}^+, b_{k+1}^-) \cdots$ .

**Case 2** The signed sequence  $T^\epsilon = \cdots (a_k^+, b_k^+)(a_{k+1}^-, b_{k+1}^-) \cdots$ .

Later on we always use “ $\cdots$ ” to denote the part that is the same as the original  $T^\epsilon$ .

*Case 1:* For the first case, by (4) we only need to consider the subcase when  $T^\epsilon = \cdots (a_k^+, b_k^+)(a_{k+1}^-, b_{k+1}^+) \cdots$ . Using commutation relation (6), we see that

$$\begin{aligned} & \langle 1', \cdots L_{a_k, b_k}^{++}(z_k, w_k) L_{a_{k+1}, b_{k+1}}^{-+}(z_{k+1}, w_{k+1}) \cdots 1 \rangle \\ &= \langle 1', \cdots L_{a_{k+1}, b_{k+1}}^{-+}(z_{k+1}, w_{k+1}) L_{a_k, b_k}^{++}(z_k, w_k) \cdots 1 \rangle \\ & \quad + \langle 1, \cdots [L_{a_k, b_k}^{++}(z_k, w_k), L_{a_{k+1}, b_{k+1}}^{-+}(z_{k+1}, w_{k+1})] \cdots 1 \rangle \\ &= \langle 1', \cdots L_{a_{k+1}, b_{k+1}}^{-+}(z_{k+1}, w_{k+1}) L_{a_k, b_k}^{++}(z_k, w_k) \cdots 1 \rangle \\ & \quad + \frac{1}{2} (b_k, a_{k+1}) (w_k - z_{k+1})^{-2} \langle 1', \cdots L_{a_k, b_{k+1}}^{++}(z_k, w_{k+1}) \cdots 1 \rangle \end{aligned}$$

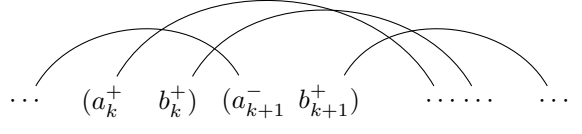
$$+ \frac{1}{2}(a_k, a_{k+1})(z_k - z_{k+1})^{-2} \langle 1' \cdots L_{b_k, b_{k+1}}^{++}(w_k, w_{k+1}) \cdots 1 \rangle. \quad (12)$$

Let

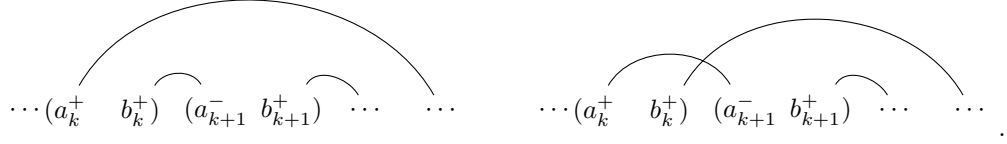
$$T_1^{\epsilon_1} = \cdots (a_{k+1}^-, b_{k+1}^+) (a_k^+, b_k^+) \cdots, \quad T_2^{\epsilon_2} = \cdots (a_k^+, b_{k+1}^+) \cdots, \quad T_3^{\epsilon_3} = \cdots (b_k^+, b_{k+1}^+) \cdots.$$

Observe that  $D(T^\epsilon)$  can be decomposed into three disjoint subsets  $D(T_1^{\epsilon_1})$ ,  $D(T_2^{\epsilon_2})$ ,  $D(T_3^{\epsilon_3})$ :

1. The case when there is no edge connecting  $(a_k^+, b_k^+)$  and  $(a_{k+1}^-, b_{k+1}^+)$ :



2. The case when there is one edge connecting  $(a_k^+, b_k^+)$  and  $(a_{k+1}^-, b_{k+1}^+)$ , and there are two subcases:



By our notation of diagram operations defined in section 3, we have:

$$D(T^\epsilon) = (D(T_1^{\epsilon_1})) \coprod (D(T_2^{\epsilon_2}) + e_1) \coprod (D(T_3^{\epsilon_3}) + e_2), \quad (13)$$

where  $e_1 = \{b_k^+, a_{k+1}^-\}$ ,  $e_2 = \{a_k^+, a_{k+1}^-\}$ .

Noting that  $T_1^{\epsilon_1} \in T^\epsilon(n, k-1)$ ,  $T_2^{\epsilon_2}, T_3^{\epsilon_3} \in T^\epsilon(n-1, k)$ , we rewrite (12), and then by the induction hypothesis we have:

$$\begin{aligned} & \langle 1', L_{a_1, b_1}^{\epsilon_1, \delta_1}(z_1, w_1), \cdots, L_{a_n, b_n}^{\epsilon_n, \delta_n}(z_n, w_n) 1 \rangle \\ &= \sum_{D \in D(T_1^{\epsilon_1})} R(D; Z, W) \\ & \quad + Q(e_1; Z, W) \sum_{D \in D(T_2^{\epsilon_2})} R(D; Z, W) + Q(e_2; Z, W) \sum_{D \in D(T_3^{\epsilon_3})} R(D; Z, W) \\ &= \sum_{D \in D(T_1^{\epsilon_1})} R(D; Z, W) \\ & \quad + \sum_{D \in D(T_2^{\epsilon_2}) + e_1} R(D; Z, W) + \sum_{D \in D(T_3^{\epsilon_3}) + e_2} R(D; Z, W) = \sum_{D \in D(T^\epsilon)} R(D; Z, W). \end{aligned}$$

So the proposition holds for this case. The third step is by observing that if  $D_1 = D - e$ , then  $c(\sigma_D) = c(\sigma_{D_1})$ , so we have

$$R(D; Z, W) = Q(e; Z, W)R(D_1; Z, W). \quad (14)$$

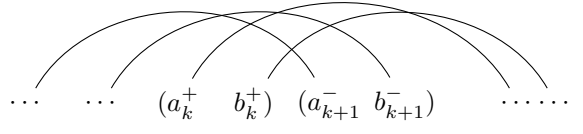
and the last step is given by (13).

*Case 2:* The arguments here are similar but there are more subcases. Formula (5) tells that

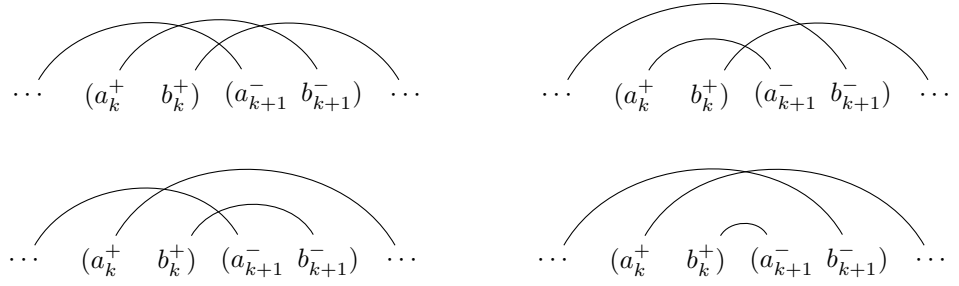
$$\begin{aligned} & \langle 1, \dots L_{a_k, b_k}^{++}(z_k, w_k) L_{a_{k+1}, b_{k+1}}^{--}(z_{k+1}, w_{k+1}) \dots 1 \rangle \\ &= \langle 1, \dots L_{a_{k+1}, b_{k+1}}^{--}(z_{k+1}, w_{k+1}) L_{a_k, b_k}^{++}(z_k, w_k) \dots 1 \rangle \\ & \quad + \langle 1, \dots [L_{a_k, b_k}^{++}(z_k, w_k), L_{a_{k+1}, b_{k+1}}^{--}(z_{k+1}, w_{k+1})] \dots 1 \rangle \\ &= \langle 1, \dots L_{a_{k+1}, b_{k+1}}^{--}(z_{k+1}, w_{k+1}) L_{a_k, b_k}^{++}(z_k, w_k) \dots 1 \rangle \\ & \quad + \frac{1}{2}(a_k, b_{k+1})(z_k - w_{k+1})^{-2} \langle 1, \dots L_{a_{k+1}, b_k}^{-+}(z_{k+1}, w_k) \dots 1 \rangle \\ & \quad + \frac{1}{2}(a_k, a_{k+1})(z_k - z_{k+1})^{-2} \langle 1, \dots L_{b_{k+1}, b_k}^{-+}(w_{k+1}, w_k) \dots 1 \rangle \\ & \quad + \frac{1}{2}(b_k, b_{k+1})(w_k - w_{k+1})^{-2} \langle 1, \dots L_{a_{k+1}, a_k}^{-+}(z_{k+1}, z_k) \dots 1 \rangle \\ & \quad + \frac{1}{2}(b_k, a_{k+1})(w_k - z_{k+1})^{-2} \langle 1, \dots L_{b_{k+1}, a_k}^{-+}(w_{k+1}, z_k) \dots 1 \rangle \\ & \quad + \frac{r}{4}(a_k, a_{k+1})(z_k - z_{k+1})^{-2}(b_k, b_{k+1})(w_k - w_{k+1})^{-2} \langle 1, \dots 1 \rangle \\ & \quad + \frac{r}{4}(a_k, b_{k+1})(z_k - w_{k+1})^{-2}(a_{k+1}, b_k)(w_k - z_{k+1})^{-2} \langle 1, \dots 1 \rangle. \end{aligned} \quad (15)$$

$D(T^\epsilon)$  can also be decomposed into three disjoint subsets:

1. The case when there is no edge connecting  $(a_k^+, b_k^+)$  and  $(a_{k+1}^-, b_{k+1}^-)$ , then we can exchange these two pairs:



2. The case when there is exactly one edge connecting  $(a_k^+, b_k^+)$  and  $(a_{k+1}^-, b_{k+1}^-)$ , and there are four subcases:



3. The case when there are exactly two edges connecting  $(a_k^+ b_k^+)$  and  $(a_{k+1}^- b_{k+1}^-)$ , and there are two subcases



Let

$$\begin{aligned} T_1^{\epsilon_1} &= \cdots (a_{k+1}^-, b_{k+1}^-) (a_k^+, b_k^+) \cdots, \quad T_3^{\epsilon_3} = \cdots \cdots \cdots \\ T_{21}^{\epsilon_{21}} &= \cdots (a_{k+1}^-, b_k^+) \cdots, \quad T_{22}^{\epsilon_{22}} = \cdots (b_{k+1}^-, b_k^+) \cdots, \\ T_{23}^{\epsilon_{23}} &= \cdots (a_{k+1}^-, a_k^+) \cdots, \quad T_{24}^{\epsilon_{24}} = \cdots (b_{k+1}^-, a_k^+) \cdots. \end{aligned}$$

and

$$e_1 = \{a_k, b_{k+1}\}, \quad e_2 = \{a_k, a_{k+1}\}, \quad e_3 = \{b_k, b_{k+1}\}, \quad e_4 = \{b_k, a_{k+1}\}.$$

Then we also have set decomposition:

$$\begin{aligned} D(T^\epsilon) &= (D(T_1^{\epsilon_1})) \coprod (D(T_{21}^{\epsilon_{21}}) + e_1) \coprod (D(T_{22}^{\epsilon_{22}}) + e_2) \coprod (D(T_{23}^{\epsilon_{23}}) + e_3) \coprod (D(T_{24}^{\epsilon_{24}}) + e_4) \coprod \\ &\quad (D(T_3^{\epsilon_3}) + e_2 + e_3) \coprod (D(T_3^{\epsilon_3}) + e_1 + e_4). \end{aligned} \quad (16)$$

Because  $T_1^{\epsilon_1} \in T^\epsilon(n, k-1)$ ,  $T_{2i}^{\epsilon_{2i}} \in T^\epsilon(n-1, k-1)$ ,  $T_3^{\epsilon_3} \in T^\epsilon(n-2, k-1)$ , we rewrite 15, by definition of  $Q(e; Z, W)$ ,  $R(D; Z, W)$ , and then by induction hypothesis we have:

$$\begin{aligned} &\langle 1', L_{a_1, b_1}^{\epsilon_1, \delta_1}(z_1, w_1), \dots, L_{a_n, b_n}^{\epsilon_n, \delta_n}(z_n, w_n) 1 \rangle \\ &= \sum_{D \in D(T_1^{\epsilon_1})} R(D; Z, W) \\ &\quad + Q(e_1; Z, W) \sum_{D \in D(T_{21}^{\epsilon_{21}})} R(D; Z, W) + Q(e_2; Z, W) \sum_{D \in D(T_{22}^{\epsilon_{22}})} R(D; Z, W) \\ &\quad + Q(e_3; Z, W) \sum_{D \in D(T_{23}^{\epsilon_{23}})} R(D; Z, W) + Q(e_4; Z, W) \sum_{D \in D(T_{24}^{\epsilon_{24}})} R(D; Z, W) \\ &\quad + (rQ(e_2; Z, W)Q(e_3; Z, W) + rQ(e_1; Z, W)Q(e_4; Z, W)) \sum_{D \in D(T_3^{\epsilon_3})} R(D; Z, W) \\ &= \sum_{D \in D(T_1^{\epsilon_1})} R(D; Z, W) + \sum_{D \in D(T_{21}^{\epsilon_{21}}) + e_1} R(D; Z, W) + \sum_{D \in D(T_{22}^{\epsilon_{22}}) + e_2} R(D; Z, W) \\ &\quad + \sum_{D \in D(T_{23}^{\epsilon_{23}}) + e_3} R(D; Z, W) + \sum_{D \in D(T_{24}^{\epsilon_{24}}) + e_4} R(D; Z, W) \\ &\quad + \sum_{D \in D(T_3^{\epsilon_3}) + e_2 + e_3} R(D; Z, W) + \sum_{D \in D(T_3^{\epsilon_3}) + e_1 + e_4} R(D; Z, W) = \sum_{D \in D(T^\epsilon)} R(D; Z, W), \end{aligned}$$

where we also use (14),(16), and noting that for  $D_2 = D - e_1 - e_2$ , then  $c(D_2) = c(D) - 1$ , and we have:

$$R(D; Z, W) = rQ(e_1; Z, W)Q(e_2; Z, W)R(D_2; Z, W). \quad (17)$$

So we conclude the proof of (11).  $\square$

Using (8) and (11) we see that

$$\begin{aligned} \langle 1', L_{a_1, b_1}(z_1, w_1) \cdots L_{a_n, b_n}(z_n, w_n) 1 \rangle &= \sum_{\epsilon} \langle 1', L_{a_1, b_1}^{\epsilon_1, \delta_1}(z_1, w_1) \cdots L_{a_n, b_n}^{\epsilon_n, \delta_n}(z_n, w_n) 1 \rangle \\ &= \sum_{\epsilon} \sum_{D \in D(T^\epsilon)} R(D; Z, W) = \sum_{D \in D(T)} R(D; Z, W). \end{aligned}$$

So we have:

**Proposition 2.**

$$\langle 1', L_{a_1, b_1}(z_1, w_1), \cdots, L_{a_n, b_n}(z_n, w_n) 1 \rangle = \sum_{D \in D(T)} R(D; Z, W). \quad (18)$$

Theorem 1 is a corollary of Proposition 2 by letting  $z_1 = w_1, \cdots, z_n = w_n$ . Noting that for  $D \in DR(T)$  we have

$$\prod_{e \in E_D} K(e, Z, Z) = \Gamma(\sigma_D; Z). \quad (19)$$

Also note that  $\forall \sigma \in DR(T)$ , we have

$$\Gamma(\sigma; T) = \sum_{D \in D(T), s.t. \sigma_D = \sigma} \Gamma(D), \quad (20)$$

by a direct computation.

Continue with (18) we have

$$\begin{aligned} \langle 1', L_{a_1, b_1}(z_1) \cdots L_{a_n, b_n}(z_n) \cdot 1 \rangle &= \sum_{\epsilon} \langle 1', L_{a_1, b_1}^{\epsilon_1, \delta_1}(z_1, w_1), \cdots, L_{a_n, b_n}^{\epsilon_n, \delta_n}(z_n, w_n) 1 \rangle \\ &= \sum_{D \in D(T)} R(D; Z, Z) \\ &= \sum_{D \in D(T)} (\Gamma(D) r^{c(\sigma_D)} \prod_{e \in E_D} K(e; Z, Z)) \\ &= \sum_{D \in D(T)} (\Gamma(D) \Gamma(\sigma_D; Z) r^{c(\sigma_D)}) \\ &= \sum_{\sigma \in DR(T)} \left( \sum_{D \in D(T), \sigma_D = \sigma} \Gamma(D) \right) \Gamma(\sigma; Z) r^{c(\sigma)} \\ &= \sum_{\sigma \in DR(T)} \Gamma(\sigma; T) \Gamma(\sigma; Z) r^{c(\sigma)}. \end{aligned}$$

where we use (10)(19) and (20). So we finally get (1) and the Theorem 1 is proved.  $\square$



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